Encoding and Decoding of Balanced q-ary Sequences Using a Gray Code Prefix

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Overview

Definition of balanced codeword

- Consider a q-ary information sequence $\mathbf{x} = x_0 x_1 x_2 \ldots x_{k-1}$, $x_i \in \{0, 1, \ldots, q-1\}$, of length k.
- Let the prefix that will be appended to $\mathbf{x}$ be of length r; and let the information and the prefix together be denoted by $\mathbf{c} = c_0 c_1 c_2 \ldots c_{k-1}$, $c_i \in \{0, 1, \ldots, q-1\}$, of length $n = k + r$.
- If $w(\mathbf{c})$ represents the weight of $\mathbf{c}$ then

$$w(\mathbf{c}) = \sum_{i=0}^{k-1} c_i.$$

- $\mathbf{c}$ is said to be balanced if

$$w(\mathbf{c}) = \frac{n(q-1)}{2} = \beta, \text{ where } \beta \text{ is an integer.}$$

Balancing of q-ary sequences

- It has been proven [1], that $\mathbf{x}$, can always be balanced by adding modulo q one sequence from a set of balancing sequences $\mathbf{b}(s,p) = b_1 b_2 \ldots b_k$ generated as follows:

$$b_i = \begin{cases} s, & i-1 \geq p, \\ s+1 \pmod{q}, & i-1 < p, \end{cases} \text{ where } \begin{cases} 0 \leq s \leq q-1, \\ 0 \leq p \leq k-1. \end{cases}$$

- Let z be the iterator through these balancing sequences, with $z = sk + p$, $0 \leq z \leq kq-1$. $\mathbf{b}(s,p)$ and $\mathbf{b}(z)$ refers to the same.
- Let $\mathbf{y}$ denote the sequence after a balancing sequence is added, $\mathbf{y} = \mathbf{x} \oplus_q \mathbf{b}(z)$. At least one $\mathbf{b}(z)$ will lead to a balanced output $\mathbf{y}$.

[1] T. G. Swart and J. H. Weber, "Efficient balancing of q-ary sequences with parallel decoding," in *Proc. IEEE Int. Symp. Inform. Theory*, Seoul, Korea, 2009.

Balancing of q-ary sequences (Cont'd)

Example (1)

For $q = 3$, $k = 3$, consider the sequence 202. The balancing value is $\beta = 3$.

z	$\mathbf{b}(z)$	$\mathbf{x} \oplus_q \mathbf{b}(z) = \mathbf{y}$	$w(\mathbf{y})$	Balanced?
0	000	$202 \oplus_3 000 = 202$	4	
1	100	$202 \oplus_3 100 = 002$	2	
2	110	$202 \oplus_3 110 = 012$	**3**	✓
3	111	$202 \oplus_3 111 = 010$	1	
4	211	$202 \oplus_3 211 = 110$	2	
5	221	$202 \oplus_3 221 = 120$	**3**	✓
6	222	$202 \oplus_3 222 = 121$	4	
7	022	$202 \oplus_3 022 = 221$	5	
8	002	$202 \oplus_3 002 = 201$	**3**	✓

q-ary Gray Codes

- Invented by Frank Gray [2]; originally used to solve problems in pulse code communication; and extended to several other fields.
- $\mathbf{d} = d_1 d_2 \dots d_{r'}$ denotes a sequence amongst the set of q-ary sequences of length r' listed in lexicographic order. They are mapped to Gray code sequences, $\mathbf{g} = g_1 g_2 \dots g_{r'}$. Any two adjacent sequences differ in only one symbol position, with weight difference of either -1 or $+1$.
- *4-ary Gray code of length 2*

z	**d**	**g**	z	**d**	**g**	z	**d**	**g**	z	**d**	**g**
0	00	00	4	10	13	8	20	20	12	30	33
1	01	01	5	11	12	9	21	21	13	31	32
2	02	02	6	12	11	10	22	22	14	32	31
3	03	03	7	13	10	11	23	23	15	33	30

[2] F. Gray, "Pulse code communication," *U. S. Patent 2632058*, 1953.

Encoding and Decoding of q-ary Gray codes [3]

Gray code encoding algorithm The parity of the sum S_i of the first $i-1$ digits of $\mathbf{g}$ determines the Gray code symbols, where $2 \leq i \leq r'$ and $g_1 = d_1$, then

$$S_i = \sum_{j=1}^{i-1} g_j, \quad \text{and} \quad g_i = \begin{cases} d_i, & \text{if } S_i \text{ is even,} \\ q-1-d_i, & \text{if } S_i \text{ is odd.} \end{cases}$$

Gray code decoding algorithm

$$S_i = \sum_{j=1}^{i-1} g_j, \quad \text{and} \quad d_i = \begin{cases} g_i, & \text{if } S_i \text{ is even,} \\ q-1-g_i, & \text{if } S_i \text{ is odd.} \end{cases}$$

[3] D.-J. Guan, "Generalized Gray codes with applications," in *Proc. National Science Council, Republic of China, Part A*, 1998.

Problem Statement

- The prefix plays an important role as it helps to decode the source information at the receiver end.
- Various previous schemes for balancing sequences were based on the assumption that a prefix can be sent [4]; without actually implementing the prefix.
- We propose an efficient, fast and easy algorithm to encode and decode balanced sequences with prefixes based on Gray codes.

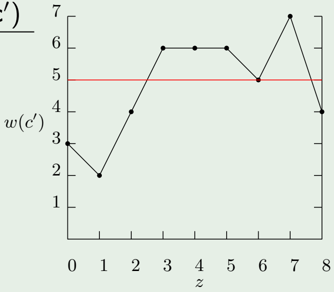
[4] T. G. Swart and J. H. Weber, "Efficient balancing of q-ary sequences with parallel decoding," in *Proc. IEEE Int. Symp. Inform. Theory*, Seoul, Korea, 2009.

Encoding

Example (2)

For $q = 3$, $k = 3$, consider the sequence 201; the condition $k = q^t$ with $t \in N$ must hold. Length of Gray code $r' = \log_q k + 1 = 2$ so $n = 5$. Graph goes through β.

z	$\mathbf{x} \oplus_q \mathbf{b}(z) = \mathbf{y}$	$\mathbf{c}' = [\mathbf{g}\|\mathbf{y}]$	$w(\mathbf{c}')$
0	$201 \oplus_3 000 = 201$	<u>00</u>201	3
1	$201 \oplus_3 100 = 001$	<u>01</u>001	2
2	$201 \oplus_3 110 = 011$	<u>02</u>011	4
3	$201 \oplus_3 111 = 012$	<u>12</u>012	6
4	$201 \oplus_3 211 = 112$	<u>11</u>112	6
5	$201 \oplus_3 221 = 122$	<u>10</u>122	6
6	$201 \oplus_3 222 = 120$	<u>20</u>120	5
7	$201 \oplus_3 022 = 220$	<u>21</u>220	7
8	$201 \oplus_3 002 = 200$	<u>22</u>200	6

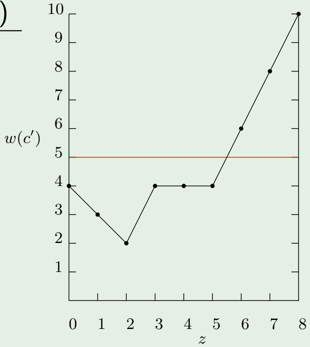


Encoding (Cont'd)

Example (3)

For $q = 3$, $k = 3$, consider the sequence 220; Graph does not go through β. Therefore we need to refine the algorithm.

z	$\mathbf{x} \oplus_q \mathbf{b}(z) = \mathbf{y}$	$\mathbf{c}' = [\mathbf{g}\|\mathbf{y}]$	$w(\mathbf{c}')$
0	$220 \oplus_3 000 = 220$	$\underline{00}220$	4
1	$220 \oplus_3 100 = 020$	$\underline{01}020$	3
2	$220 \oplus_3 110 = 000$	$\underline{02}000$	2
3	$220 \oplus_3 111 = 001$	$\underline{12}001$	4
4	$220 \oplus_3 211 = 101$	$\underline{11}101$	4
5	$220 \oplus_3 221 = 111$	$\underline{10}111$	4
6	$220 \oplus_3 222 = 112$	$\underline{20}112$	6
7	$220 \oplus_3 022 = 212$	$\underline{21}212$	8
8	$220 \oplus_3 002 = 222$	$\underline{22}222$	10

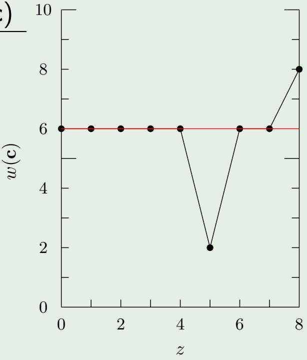


Encoding (Cont'd)

Example (Encoding)

For $q = 3$, $k = 3$, consider the sequence 212; Extra digit $u \in [0, q-1]$ and $u = \beta - w(\mathbf{c}')$.

z	$\mathbf{x} \oplus_q \mathbf{b}(z) = \mathbf{y}$	$\mathbf{c} = [u\|\mathbf{g}\|\mathbf{y}]$	$w(\mathbf{c})$
0	$212 \oplus_3 000 = 212$	100212	6
1	$212 \oplus_3 100 = 012$	201012	6
2	$212 \oplus_3 110 = 022$	002022	6
3	$212 \oplus_3 111 = 020$	112020	6
4	$212 \oplus_3 211 = 120$	111120	6
5	$212 \oplus_3 221 = 100$	010100	2
6	$212 \oplus_3 222 = 101$	220101	6
7	$212 \oplus_3 022 = 201$	021201	6
8	$212 \oplus_3 002 = 211$	022211	8



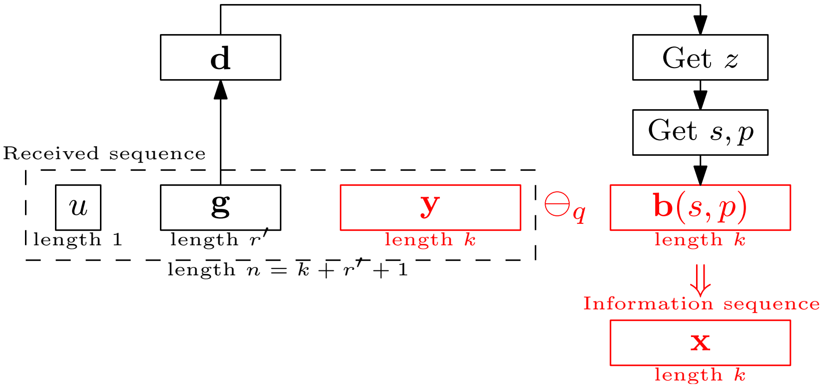
Encoding (Cont'd)

Encoding algorithm: Balance the sequence by finding the correct Gray code prefix:

1. Incrementing through z, determine the balancing sequence $\mathbf{b}(s, p)$ and add it to the information sequence $\mathbf{x}$ to obtain $\mathbf{y}$.
2. For each increment, convert z into base q over r' symbols and determine the corresponding Gray code sequence, $\mathbf{g}$.
3. Set $u = \beta - w(\mathbf{c}')$, provided that $u \in \{0, 1, \ldots, q-1\}$, otherwise set $u = 0$.
4. Continue incrementing z until the weight of u, $\mathbf{g}$ and $\mathbf{y}$ together is equal to β.

Decoding

The figure below shows the decoding process.



Decoding

Example (Decoding)

Consider the case $q = 3$, $n = 13$, where a sequence was encoded as 1012000122022, with a $(3, 3)$-Gray code.

- The first symbol 1 is dropped, then the Gray code sequence is extracted as 012, and decoded as 010.
- Thus $z = 3$, leading to $s = 0$ and $p = 3$, and $\mathbf{b}(0, 3) = 111000000$.
- Finally, the information sequence is recovered as $\mathbf{x} = \mathbf{y} \ominus_q \mathbf{b}(s, p) = 000122022 \ominus_3 111000000 = 222122022.$

Redundancy

The information sequence length k as a function of r for different constructions.

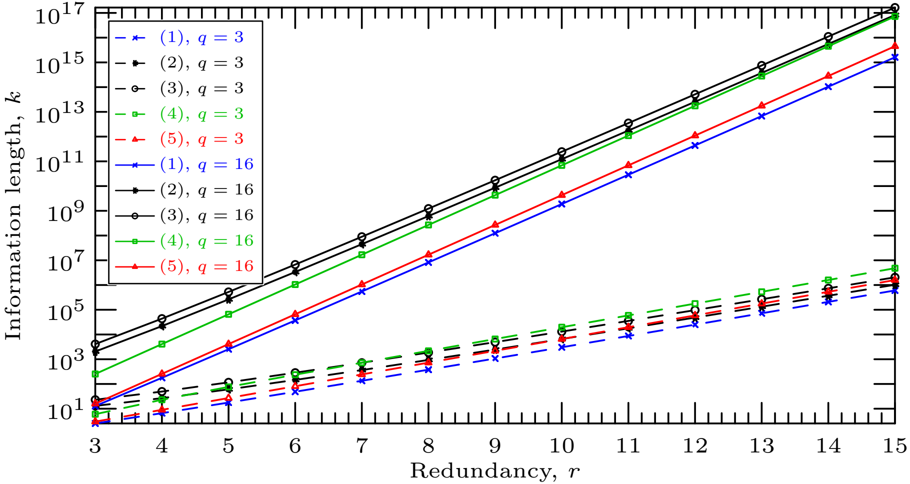
- (1) $k \leq q^{r-1}\sqrt{\frac{6}{\pi r(q^2-1)}}$ [9]
- (2) $k \leq \frac{q^r-1}{r-1}$ [10]
- (3) $k \leq 2\frac{q^r-1}{r-1} - r$ [10]
- (4) $k = q^{r-1} - r$ [11]
- (5) $k = q^{r-2}$ (This is our construction.)

[9] T. G. Swart and J. H. Weber, "Efficient balancing of q-ary sequences with parallel decoding," in *Proc. IEEE Int. Symp. Inform. Theory*, Seoul, Korea, 2009.

[10] R. M. Capocelli, L. Gargano and U. Vaccaro, "Efficient q-ary immutable codes," *Discrete Applied Mathematics*, 1991.

[11] L. G. Tallini and U. Vaccaro, "Efficient m-ary immutable codes," *Discrete Applied Mathematics*, 1999.

Redundancy


10^{17}
10^{15}
10^{13}
10^{11}
10^{9}
10^{7}
10^{5}
10^{3}
10^{1}
Information length, k
(1), $q = 3$
(2), $q = 3$
(3), $q = 3$
(4), $q = 3$
(5), $q = 3$
(1), $q = 16$
(2), $q = 16$
(3), $q = 16$
(4), $q = 16$
(5), $q = 16$
3 4 5 6 7 8 9 10 11 12 13 14 15
Redundancy, r

Complexity

- Previous schemes discussed in [12] and [13] require $\mathcal{O}(qk \log_q k)$ digit operations for the encoding and decoding processes.
- The scheme in [14] requires $\mathcal{O}(qk \log_q k)$ digit operations for the encoding and $\mathcal{O}(k)$ digit operations for the decoding.
- Our decoding process requires $\mathcal{O}(k + \log_q k)$ digit operations. Fast parallel decoding, after $\mathbf{b}(s, p)$ has been determined from the Gray code.
- Encoding takes longer than decoding. In the worst case where the kq-th balancing sequence and Gray code result in balancing, $\mathcal{O}(qk \log_q k)$ digit operations are needed.

[12] R. M. Capocelli, L. Gargano and U. Vaccaro, "Efficient q-ary immutable codes," *Discrete Applied Mathematics*, vol. 33, 1991.

[13] L. G. Tallini and U. Vaccaro, "Efficient m-ary immutable codes," *Discrete Applied Mathematics*, 1999.

[14] T. G. Swart and J. H. Weber, "Efficient balancing of q-ary sequences with parallel decoding," in *Proc. IEEE Int. Symp. Inform. Theory*, Seoul, Korea, 2009.

Conclusion

- A simple algorithm was presented to encode and decode balanced q-ary information sequences of length k, where $k = q^t$. This is based on a Gray code prefix that encodes the balancing index.
- Both the balancing and Gray code algorithms are efficient as only simple addition and subtraction operations are used, and no lookup tables are needed.
- The majority of the decoding algorithm can also be performed in parallel.
- As future work, this algorithm will be extended to q-ary sequences of length k, where $k \neq q^t$.
- An investigation into whether the extra symbol u can be eliminated for certain values of k and q has to be explored.

Thanks for your attention!

"We cannot solve our problems with the same thinking we used when we created them." Albert Einstein

QUESTIONS AND COMMENTS